Problem Set 6 due October 21, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Consider the matrix $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 5 & 1 \\ 0 & -2 & 2 \end{bmatrix}$.

(1) Compute the projection matrices P_C and P_R onto the column and row spaces of A, respectively. *(10 points)*

(2) Compute $P_C A$ and AP_R and give a geometric explanation of your answer. *(10 points)*

Problem 2: (1) Orthogonal matrices have the property that $Q^T Q = I$. Prove that the product of two general orthogonal matrices Q_1 and Q_2 is an orthogonal matrix. *(10 points)*

(2) Suppose you have non-zero mutually orthogonal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$. Prove that they must be linearly independent. *(10 points)*

Problem 3: (1) Use Gram-Schmidt to compute an orthonormal basis of $\mathbb{R}^3$ that includes the vector $\boldsymbol{q}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. *(10 points)*

(2) Compute the $A = QR$ factorization of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

(where Q has orthonormal columns and R is square upper triangular). *(10 points)*

Problem 4: Consider a length 1 vector $\boldsymbol{a} \in \mathbb{R}^n$ (so $||\boldsymbol{a}|| = 1$), and look at the linear transformation:

$$\phi : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{corresponding to the matrix} \qquad A = I - 2\boldsymbol{a}\boldsymbol{a}^T$$

(1) Compute $\boldsymbol{a}^T \boldsymbol{a}$ and show that the matrix A is orthogonal. *(10 points)*

(2) What is the subspace of $\mathbb{R}^n$ fixed by ϕ, i.e. the subspace:

$$\{\boldsymbol{v} \in \mathbb{R}^n \text{ such that } \phi(\boldsymbol{v}) = \boldsymbol{v}\}$$

(5 points)

(3) Compute $\phi(\boldsymbol{a})$ and describe the linear transformation ϕ geometrically (i.e. say what it is called in plain English, and draw a picture in the $n = 3$ case). *(10 points)*

Problem 5: Consider the function:

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y + 2 \\ 3x + y - 2 \end{bmatrix}$$

(1) Explain why f is <u>not</u> a linear transformation. *(5 points)*

(2) Find a linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and translations $\sigma, \tau : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

$$f = \sigma \circ \phi \qquad \text{and} \qquad f = \phi \circ \tau$$

(a translation is a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $g(\boldsymbol{v}) = \boldsymbol{v} + \boldsymbol{a}$ for a fixed vector $\boldsymbol{a}$). *(10 points)*